

THE THERMAL-WAVE MODEL: A SCHRÖDINGER-LIKE EQUATION FOR CHARGED PARTICLE BEAM DYNAMICS ¹

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Abstract

In this paper we review some results on longitudinal beam dynamics obtained in the framework of the *Thermal Wave Model* (TWM). In this model, which has recently shown the capability to describe both longitudinal and transverse dynamics of charged particle beams, the beam dynamics is ruled by Schrödinger-like equations for the beam-wave-functions, whose squared modulus is proportional to the beam density profile. Remarkably, the role of the Planck constant is played by a diffractive constant ϵ , the emittance, which has a thermal nature.

1 Introduction

Recently, on pure basis of analogy with other similar subjects of physics, a new technique to derive the equation of motion for a thermal system, like a charged particle beam at finite temperature, which is able to take into account the collective behaviour of the ensemble has been obtained [1]-[5].

The starting point of this technique, the *Thermal Wave Quantization* (TWQ), are the equations of motion of the considered system, in the so called *single-particle approximation*, from which is possible to obtain the *single-particle hamiltonian* of the system. At this point, the formal analogy showed in the case of the transverse dynamics for relativistic charged particle beams, with the electromagnetic optics in paraxial approximation and with the two dimensional nonrelativistic quantum mechanics [1], suggests to replace the *single-particle hamiltonian*, with a differential operator, and the hamilton-equations with a Schrödinger-like equation, in which coordinate and particle momentum are replaced by a beam-wave-function.

This technique, applied to the longitudinal and transverse beam dynamics, has led to the formulation of the *Thermal Wave Model* (TWM) for relativistic charged particle beam propagation, which represents a useful *quantum-like* description of the total beam optics [1]-[5]. This model has already been successfully applied for estimating the effects of the aberrations in linear colliders [3], [5], as well as for describing nonlinear beam-plasma interaction [2], and nonlinear longitudinal dynamics in circular accelerating machines [4].

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2 TWM for longitudinal dynamics

Let us consider a single relativistic particle of electric charge q , within a stationary bunch, travelling with longitudinal velocity βc ($\beta \simeq 1$) in a circular accelerating machine of radius $R_0 = cT_0/2\pi$ (T_0 being the revolution period). Its longitudinal motion is described, neglecting radiation damping and quantum excitations, by a pair of equations which, defining $s \equiv ct$ (t being the time), can be put in the following dimensionless form [6]

$$\frac{dx}{ds} = \eta \frac{\Delta E}{E_0} \equiv \eta \mathcal{P} \quad , \quad (1)$$

$$\frac{d\mathcal{P}}{ds} = -\frac{q\Delta V}{cT_0 E_0} \quad , \quad (2)$$

where x is the longitudinal particle coordinate and $\mathcal{P} \equiv \Delta E/E_0$ is the dimensionless longitudinal energy variation, after a turn in the ring. Note that x ($-\pi R_0 \leq x \leq \pi R_0$) and \mathcal{P} are both evaluated with respect to the synchronous particle ($\Delta E = 0$), and E_0 is the synchronous particle energy. The quantity ΔV represents the total voltage variation after a turn and it takes into account the interactions of the particles with the surrounding medium (RF-cavities, pipe, kickers, etc.). Consequently, the equations (1) and (2) describe the longitudinal bunch dynamics on time scale $t \gg T_0$. Furthermore, in (1) the parameter η is defined as $\eta \equiv (\Delta\omega/\omega_0)/(\Delta E/E_0)$ ($\omega_0 \equiv c/R_0$ and $\Delta\omega$ being the frequency shift with respect to ω_0). By defining the *momentum compaction* $\alpha \equiv (\Delta R/R_0)/(\Delta E/E_0)$, where ΔR is the orbit radius variation with respect to R_0 , it can be easily proved that $\eta = 1/\gamma^2 - \alpha$. From (1) and (2) we can easily write the following dimensionless *single-particle hamiltonian*

$$H = \frac{1}{2}\eta\mathcal{P}^2 + U \quad , \quad (3)$$

where

$$U \equiv \frac{1}{cT_0 E_0} \int_0^x q\Delta V \, dx' \quad . \quad (4)$$

In order to find an equation which describes the longitudinal evolution of the beam, taking into account its thermal spreading (longitudinal emittance) while it interacts with the surrounding medium (potential well and wake fields), we follow the *quantum-analogy*, which suggests to use in (3) the following correspondence rules

$$\mathcal{P} \rightarrow \hat{\mathcal{P}} \equiv -i\epsilon_L \frac{\partial}{\partial x} \quad \text{and} \quad H \rightarrow \hat{H} \equiv i\epsilon_L \frac{\partial}{\partial s} \quad , \quad (5)$$

where ϵ_L is the longitudinal beam emittance. Thus, by considering (3) and (5), the following Schrödinger-like equation for the *beam wave function* (bwf) Ψ can be assumed

$$i\epsilon_L \frac{\partial \Psi}{\partial s} = \hat{H} \Psi \quad , \quad (6)$$

where $\hat{H} = \frac{1}{2}\hat{\mathcal{P}}^2 + U$. Consequently, (6) becomes

$$i\epsilon_L \frac{\partial \Psi}{\partial s} = -\frac{\epsilon_L^2 \eta}{2} \frac{\partial^2 \Psi}{\partial x^2} + U \Psi \quad . \quad (7)$$

Note that (7) describes the longitudinal beam dynamics in terms of the bwf Ψ , which we assume to be related to the longitudinal density number $\lambda(x, s)$ through the following relation: $\lambda(x, s) = \lambda_0 |\Psi(x, s)|^2$, where $\lambda_0 = N/R_0$ (N being the total number of particle in a bunch). According to the previous definitions, $|\Psi|^2$ gives the longitudinal beam density profile. Furthermore, the circular topology of the ring should requires periodic solutions for Ψ , with respect to x ($\Psi(\pi R_0, s) = \Psi(-\pi R_0, s)$ and $\partial_x \Psi(\pi R_0, s) = \partial_x \Psi(-\pi R_0, s)$). In these conditions, from (7) the norm squared (\mathcal{N}^2) of bwf, defined as

$$\mathcal{N}^2 \equiv \int_{-\pi R_0}^{+\pi R_0} |\Psi(x, s)|^2 dx \quad , \quad (8)$$

is conserved (U is assumed a real function), and it has been fixed for simplicity equal to R_0 . This result is compatible with the physical requirement that $\int_{-\pi R_0}^{+\pi R_0} \lambda(x, s) dx = N$. However, in the following we restrict our analysis to consider bunched beam whose effective length is much smaller than R_0 . Under this assumption, the above conditions of periodicity of Ψ do not have a relevant role, because in this limit, for the bunch, the ring looks like an infinite linear accelerator. Thus we can define the effective bunch length σ_L , and the expectation value of the momentum \hat{P} as

$$\sigma_L(s) \equiv \left[\frac{1}{\mathcal{N}^2} \int_{-\pi R_0}^{+\pi R_0} x^2 |\Psi|^2 dx \right]^{1/2} \quad \sigma_L^p(s) \equiv \frac{\epsilon_L}{\mathcal{N}} \left[\int_{-\pi R_0}^{+\pi R_0} \left| \frac{\partial \Psi}{\partial x} \right|^2 dx \right]^{1/2} \quad , \quad (9)$$

and in complete analogy to quantum mechanics the *uncertainty principle*

$$\sigma_L \sigma_L^p \geq \epsilon_L/2 \quad (10)$$

holds. Furthermore, note that (7) $1/\eta$ plays the role of an *effective mass*.

2.1 The interaction potential and synchrotron oscillations

As it has been shown in Ref. [4], the potential U can be split in two parts (RF + Self-interaction) and (4) becomes: $U = U_{RF} + U_S$. Note that whereas in general U_{RF} is a known function of x and s , the explicit expression for U_S , depending on the bunch density (collective effects), needs particular assumptions about the beam interaction with the surrounding medium to be specified. This interaction can be parametrized in terms of the longitudinal coupling impedance [7]. In the special case of a linear approximation for the RF-potential, $U_{RF} = \frac{1}{2} \frac{K}{\eta} x^2$, where K is the *cavity strength* ($\sqrt{|K|}$ is the synchrotron wave number), and for a purely reactive longitudinal coupling impedance X , the equation for bwf becomes

$$i\epsilon_L \eta \frac{\partial \Psi}{\partial s} = -\frac{\epsilon_L^2 \eta^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2} K x^2 \Psi - \eta \frac{qI}{2\pi E_0} \left(\frac{X}{n} \right) |\Psi|^2 \Psi \quad , \quad (11)$$

where I is the beam current, and n is the so-called *harmonic number* [8]. In the simplest case of ($X = 0$) and for $R_0 \gg \sigma_L$, the Eq. (11) can be exactly solved, and the normalized solutions for bwf are the well known Hermite-Gauss modes as it occurs in complete analogy for electromagnetic optics in paraxial approximation [9]

$$\Psi_m(x, s) = \frac{\exp \left[-\frac{x^2}{4\sigma_L^2(s)} \right]}{[2\pi\sigma_L^2(s)]^{1/4}} H_m \left(\frac{x}{\sqrt{2}\sigma_L(s)} \right) \exp \left[i \frac{x^2}{2\epsilon_L \eta \rho_L(s)} + i(1 + 2m)\phi_L(s) \right] \quad . \quad (12)$$

In (12) the functions $\sigma_L(s)$, $\rho_L(s)$ and $\phi_L(s)$ are solutions of the following system of differential equations

$$\frac{d^2\sigma_L}{ds^2} + K\sigma_L - \frac{\epsilon_L^2\eta^2}{4\sigma_L^3} = 0 \quad , \quad (13)$$

$$\frac{1}{\rho_L} = \frac{1}{\sigma_L} \frac{d\sigma_L}{ds} \quad , \quad (14)$$

$$\frac{d\phi_L}{ds} = -\frac{\epsilon_L\eta}{4\sigma_L^2} \quad . \quad (15)$$

and $H_m(x)$ are the Hermite-polynomials with m a non-negative integer. Note that $|\Psi_m|^2$ for $m = 0$ (fundamental mode) gives a Gaussian particle distribution. Remarkably, it can be easily proved that (13) is completely equivalent to

$$\frac{d^2\sigma_L^2}{ds^2} + 4K\sigma_L^2 = 4\mathcal{E} \quad \text{with} \quad \mathcal{E} = \frac{1}{2} \left(\frac{d\sigma_L}{ds} \right)^2 + \frac{\epsilon_L^2\eta^2}{8\sigma_L^2} + \frac{1}{2}K\sigma_L^2 = \text{const.} \quad (16)$$

In this form it is easily to recognize that the equation for $\sigma_L(s)$ (16), i.e. (13), describes the synchrotron oscillations. The equilibrium condition $d^2\sigma_L/ds^2 = 0$ gives

$$\sigma_L^0 = \frac{|\eta|}{\sqrt{|K|}} \sigma_L^{p0} = \frac{|\eta|R}{\nu_s} \sigma_L^{p0} \quad , \quad (17)$$

where σ_L^0 and σ_L^{p0} are the equilibrium value of σ_L and σ_L^p , respectively, and ν_s is the synchrotron number given by the ratio between the synchrotron frequency $\Omega_s \equiv c\sqrt{|K|}$ and the revolution frequency ω_0 [10]. Equation (17) recovers the well known relationship between the bunch length σ_L^0 and the energy spread σ_L^{p0} [10]. Since for the present case the bwf is Gaussian, in obtaining (17) we have introduced the minimum value $\sigma_L^0\sigma_L^{p0} = \epsilon_L/2$ of the product $\sigma_L\sigma_L^p$ consistently with disequality reported in section 2.

3 Coherent stability criterion and soliton solution

In this Section, we develop, within the framework of the thermal wave model, an analysis of some collective effects occurring when the bunch interacts with the surrounding medium. To this end, we consider the special case of RF cavity *off* and take into account both the space charge effect and a purely inductive coupling impedance. Consequently, (11) becomes

$$i\epsilon_L\eta\frac{\partial\Psi}{\partial s} = -\frac{\epsilon_L^2\eta^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \eta\frac{qI}{2\pi E_0}\left(\frac{X}{n}\right)|\Psi|^2\Psi \quad . \quad (18)$$

Note that (18) is formally identical to the cubic NLS equation which describes the propagation of an e.m. pulse through a nonlinear medium in paraxial approximation [11],[12],[13]. In this analogy, the factor $\epsilon_L\eta$ plays the role of the diffraction parameter (the inverse of the wave number), s corresponds to the time, and $-\eta qI(X/n)/(2\pi E_0)|\Psi|^2$ corresponds to a nonlinear refractive index. Thus, an analysis of the bunch coherent instability (stability) can be made in complete analogy to the electromagnetic case [11],[12]. To this aim, applying the well known method developed in

nonlinear e.m. optics to search for the sufficient conditions of modulational instability, we show that coherent instability for particle bunches is fully equivalent to modulational instability for e.m. bunches. Moreover, a soliton-envelope solution, very interesting for accelerator physics, is found.

As an example, we analyze the instability of a plane-wave ($\Psi_0(x, s) = \rho_0 \exp[i(k_0 x - \Omega_0 s)]$, where ρ_0 is a positive constant) solution of Eq. (18), when a small perturbation around it is introduced. Let

$$\Psi(x, s) = [\rho_0 + \rho_1(x, s)] \exp[i(k_0 x - \Omega_0 s) + i\theta_1(x, s)] \quad (19)$$

be the perturbed solution, being ρ_1 and θ_1 real functions, and

$$\Omega_0 = \frac{\epsilon_L \eta}{2} k_0^2 - \frac{qI}{2\pi\epsilon_L E_0} \left(\frac{X}{n}\right) \rho_0^2 \quad (20)$$

In order to obtain the dispersion relation of the system we can assume

$$\rho_1(x, s) = \rho_1^0 \cos(kx - \Omega s + \phi_0) \quad \theta_1(x, s) = \theta_1^0 \sin(kx - \Omega s + \phi_0) \quad , \quad (21)$$

where ρ_1^0 , θ_1^0 , k , Ω and ϕ_0 are real constants. By imposing that (19) is a solution of the linearized (18) we obtain the following dispersion relation

$$\Omega = \frac{\epsilon_L \eta}{2} k \left[2k_0 \pm \sqrt{\left(k^2 - \frac{4qI}{\eta\epsilon_L^2 2\pi E_0} \left(\frac{X}{n}\right) \rho_0^2 \right)} \right] \quad (22)$$

Reminding that unstable modes occur for $\Im(\Omega) \neq 0$, namely

$$k^2 < \frac{2qI}{\pi\eta\epsilon_L^2 E_0} \left(\frac{X}{n}\right) \rho_0^2 \quad , \quad (23)$$

we get stability for $\eta X < 0$ and instability for $\eta X > 0$. This result recovers the well-known condition for coherent stability (instability) for monochromatic charged particle beams [14], in addition we remark that the above condition is fully similar to the Lighthill criterion, valid for modulational instability of an e.m. plane-wave travelling in a nonlinear medium [11],[12],[13].

For a bunched beam ($\sigma_L \ll R_0$), a solitary solution of (18) is found by looking for a solution of a relativistic envelope form:

$$\Psi(x, s) = G(x - \beta_0 s) e^{i\chi_0 x - i w_0 s} \quad , \quad (24)$$

with χ_0 , w_0 , and β_0 real numbers. Thus, according to the general theory of NLS equations [11], the following soliton-like solution for the beam density ($\lambda = \lambda_0 G^2$), which satisfies (8), is possible under the condition $\eta X > 0$:

$$\lambda(x, s) = \frac{N^2 q^2 R}{4\epsilon_L^2 T_0 E_0} \frac{1}{\eta} \left(\frac{X}{n}\right) \text{sech}^2 \left[\frac{N q^2 R}{2\epsilon_L^2 T_0 E_0} \frac{1}{\eta} \left(\frac{X}{n}\right) (x - \beta_0 s) \right] \quad , \quad (25)$$

where

$$\chi_0 = \frac{\beta_0}{\epsilon_L \eta} \quad w_0 = \frac{\epsilon_L \eta}{2} \chi_0^2 - \frac{1}{2\eta\epsilon_L} \left[\frac{N q^2 R}{2\epsilon_L T_0 E_0} \left(\frac{X}{n}\right) \right]^2 \quad (26)$$

4 Conclusions

In this paper, we have reviewed some results and applications of the *Thermal Wave Model*, showing in particular, how it is possible to give a novel approach to the study of the nonlinear longitudinal dynamics of a relativistic particle bunch in circular accelerating machines. Neglecting radiation damping and quantum excitation, we have shown that the nonlinear interaction between the bunch and the surroundings (potential well and wake fields) is governed by an appropriate NLS equation (equation (11)), fully similar to the one that holds for the propagation of an e.m. bunch in a nonlinear medium in paraxial approximation [11], [12],[13]. Much remains to be done — like, for instance, the extension to 2- or even to 3-D, or the development of an iterable formulation — to make this model really interesting for the study of the typical, still unsolved, beam-dynamics problems. Nevertheless, its very innovative feature of allowing the treatment of the whole beam at the same time, makes it look extremely promising for a new, and more complete approach to the subject.

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